

## Seidel Laplacian Energy of Fuzzy graphs

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### Abstract

The energy of a graph is related to its spectrum, which is equal to the total of the latent values of the pertinent adjacency matrix. In this research work, we proposed some of the features and the energy of the Seidel Laplacian of a fuzzy graph. Also, the lower and upper bounds for the energy of the Seidel Laplacian of a fuzzy graph were studied with suitable illustrative examples.

**Keywords:** Graph, Energy of a graph, Seidel Laplacian energy, Fuzzy set, Fuzzy graph

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### 1. Introduction

Many difficulties in daily life are solved using graph theory. For instance, Euler resolved the infamous "Konigsberg bridges" problem in 1735. Although it is the first platform to employ concepts from graph theory, it is difficult to glean precise information about it due to the system's size and complexity [1]. Fuzzy graphs are used in such systems to solve them.

Zadeh proposed fuzzy sets in 1965, whose concept is only a membership function. Based on his fuzzy relationship, Kaufman suggested the fuzzy graph in 1973. Fuzzy sets were used in real-life problems with uncertainty. Fuzzy logic can be used in many applications, like tracking the maximum power from solar power voltaic and controller circuit design applications [2–5].

Later, Rosenfeld elaborated on the definition of graph theory, which includes fuzzy vertices and fuzzy edges. One of the most current research areas in what has become a promising multidisciplinary field is fuzzy graph theory. Graph theory depicts a group of items and their

relationships through pictures or diagrams. There are numerous applications for graph theory in different categories, such as Mathematics, Information technology, Computer science, Chemistry, Modelling, Networking, Physics and Biology to mention a few.

Now a days, a greater number of works are involved in some particular topics like labelling, indices, and hubset [6–8]. Akalyadevi et al. defined the Spherical neutrosophic cubic fuzzy models in a bipolar environment and studied their operations [9–12]. The pair of sets (V, E) that represent the formal definition of a graph are V, which is the set of nodes, and E, which is the set of edges linking the vertices. Ivan Gutman initiated the energy concept in 1978. The energy of the graph is calculated using the eigen (latent) values of the relevant adjacency matrix. A graph's overall absolute value of its latent values as determined by its adjacency matrix. McClelland also investigated the graph's energy boundaries, for which he developed the McClelland inequality. For fuzzy graphs, Sunil Mathew et al. defined the adjacency matrix and energy. Also, some conclusions concerning the energy bounds for weighted and simple graphs are enhanced.

Fuzziness is added to energy calculations to increase their resilience, adaptability, and

comprehensiveness, which eventually results in more efficient and sustainable energy management. The relevance of applying energy ideas to fuzzy graphs is that it enables a more precise, thorough, and realistic description of complex systems with uncertain or imprecise interactions. This allows to learn more about the dynamics, behaviour and optimisation of these systems across a variety of academic disciplines. The concept of a graph's energy is expanded to include a fuzzy graph's energy. For further study, refer to [13-21]. Here the concept of a Seidel Laplacian (Sl) graph's energy is expanded to a fuzzy graph's energy for Seidel Laplacian.

Table 1. Notations

S.No	Symbols	Notations
1	$\zeta$	fuzzy graph
2	$\Gamma$	fuzzy set
3	$l(\zeta)$ , $le(\zeta)$	Laplacian matrix, Laplacian energy
4	$E_i$	Latent values of Seidel Laplacian matrix
5	$d_i$	$i$ th node's degree
6	$S(\zeta)$	Seidel matrix
7	$Sl(\zeta)$	Seidel Laplacian matrix
8	$Sle(\zeta)$	Energy of Seidel Laplacian matrix
9	$d(\zeta)$	diagonal matrix of node degrees

## 2. Preliminaries

Throughout this article, consider  $\zeta$  is a fuzzy graph with 'q' edges and 'p' nodes with an order 'p' matrix.

### Definition :2.1[2]

A fuzzy set 'T' of a set is defined as a function, and the value of T describes a degree of membership for all T in X.

### Definition :2.2[18]

$\zeta = (\vartheta, \Gamma, \sigma)$  be a fuzzy graph that consists of a non-empty set  $\vartheta$  and two functions  $\Gamma: \vartheta \rightarrow [0,1]$ ,  $\sigma: \vartheta * \vartheta \rightarrow [0,1]$  such that for all  $x, y$  in  $\vartheta$ ,  $\sigma(x, y) \leq \Gamma(x) \cap \Gamma(y)$ .  $\Gamma$  is the fuzzy node set and  $\sigma$  is the fuzzy edge set on the graph  $\zeta$ .  $\Gamma$  is the fuzzy relation on  $\sigma$ .

The latent value of the adjacency matrix, also known as the latent (eigen) value of a graph, is represented by the letters  $e_1, e_2, e_3$ , and so on. A graph's energy is the total of its latent values' absolute values.

### Result:2.3

The following relations were met by the eigenvalues of the ordinary and Laplacian matrices:

- (i) In the adjacency matrix, the latent values added together equal 0.
- (ii) The adjacency matrix's latent value square sum equals 2 \* the number of edges.
- (iii) The entire sum of the latent values of the Laplacian matrix is equal to 2 times the number of edges.
- (iv) The laplacian matrix's total latent values are calculated as (2\* number of edges) + (sum of squares of all degrees).

### Definition: 2.4[15, 17]

The Laplacian matrix  $\zeta$  is defined as  $l(\zeta) = d(\zeta) - a(\zeta)$ . Let  $e_1, e_2, \dots, e_p$  be its latent values. Then the Laplacian energy of  $\zeta$  is defined as  $le(\zeta) = \sum_{i=1}^p \left| e_i - \frac{2 \sum a_{ij}}{p} \right|$ .

### Definition:2.5[17]

The Seidel matrix of a graph is the real symmetric matrix  $S(\zeta) = (s_{ij})$ , where  $s_{ij} = 1$  for adjacent vertices and  $s_{ij} = -1$  for non-adjacent vertices and  $s_{ij} = 0$  for nodes with  $i = j$ .

### Definition:2.6[17]

$$dS(\zeta) = \text{diagonal}(p - 1 - 2d_i) \text{ where } i=1, 2, \dots, n.$$

### Definition: 2.7[17]

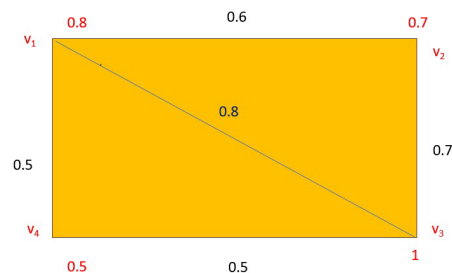
The Sl matrix of a fuzzy graph  $\zeta$  as defined  $Sl(\zeta) = dS(\zeta) - S(\zeta)$ .

### Definition: 2.8[15, 17]

Let  $E_1, E_2, E_3, \dots, E_p$  be the latent values of  $Sl(\zeta)$  and define Sl energy of a fuzzy graph  $\zeta$  as  $Sle(\zeta) = \sum_{i=1}^p \left| E_i - \frac{p(p-1) - 4 \sum a_{ij}}{p} \right|$  where  $1 \leq i < j \leq p$  and denote  $\varphi_i = E_i - \frac{p(p-1) - 4 \sum a_{ij}}{p}$ .

## 3. Seidel Laplacian Energy of fuzzy graph

**Example 3.1** In Fig 1, consider the following fuzzy graph  $\zeta$  with '5' edges and '4' nodes.



**Figure 1.**  $\zeta$  – A Fuzzy graph

Then  $d(\zeta)$ ,  $dS(\zeta)$  and  $S(\zeta)$  is defined as follows

$$d(\zeta) = \begin{pmatrix} 1.9 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$dS(\zeta) = \begin{pmatrix} -0.8 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{and } S(\zeta) = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

The  $Sl$  matrix of  $\zeta$  is defined as

$$Sl(\zeta) = dS(\zeta) - S(\zeta)$$

$$= \begin{pmatrix} -0.8 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -0.8 & 1 & 1 & 1 \\ 1 & 0.4 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

The latent values of  $Sl(\zeta)$  is  $E_i$  are 1.58818, -1.8829, 2.03624 and -2.14152.

The Seidel Laplacian energy of a fuzzy graph  $\zeta$  is

$$Sle(\zeta) = \sum_{i=1}^p \left| E_i - \frac{p(p-1) - 4 \sum q_{ij}}{p} \right|$$

$$= \sum_{i=1}^p \left| E_i - \frac{4(4-1) - 4(3.1)}{4} \right|$$

$$= \sum_{i=1}^p |E_i - (-0.1)|$$

$$= |1.58818+0.1| + |-1.8829+0.1| + |2.03624+0.1| + |-2.14152+0.1| = 7.64884.$$

**Theorem 3.2**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph. The following relations are therefore satisfied by the latent values of the  $Sl$  matrix.

$$(i) \sum_{i=1}^p E_i = (p^2 - p) - 4 \sum q_{ij}$$

$$(ii) \sum_{i=1}^p E_i^2 = p^2(p-1) - 8(n-1) \sum q_{ij} + 4 \sum d_i^2$$

**Proof:**

$$(i) \sum_{i=1}^p E_i = \text{trace}(Sl(\zeta)) = \sum_{i=1}^p p - 1 - 2d_i = (p^2 - p) - 4 \sum q_{ij}$$

$$(ii) \sum_{i=1}^p E_i^2 = \text{trace}(Sl(\zeta))^2$$

$$= p^2(p-1) - 8(p-1) \sum q_{ij} + 4 \sum d_i^2$$

Since  $\sum_{i=1}^p E_i = \text{trace}(Sl(\zeta)) =$  The sum of the latent values of  $Sl(\zeta)$

$$= 1.58818 - 1.8829 + 2.03624 - 2.14152 = -$$

$$0.4$$

$$\text{and } p(p-1) - 4 \sum q_{ij} = 4(4-1) - 4(3.1) = -0.4.$$

Checking the theorem using example 3.1,

$$\sum_{i=1}^p E_i^2 = \text{trace}(Sl(\zeta))^2 = \text{the squares of the latent values added together} = 14.8$$

and

$$p^2(p-1) - 8(p-1) \sum q_{ij} + 4 \sum d_i^2 = 16(3) - 8(3)(3.1) + 4(10.3) = 14.8.$$

**Theorem 3.3**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph. Then  $\sum_{i=1}^p \varphi_i = 0$

**Proof:**

$$\text{Since } \sum_{i=1}^p \varphi_i = \sum_{i=1}^p E_i - \frac{(p^2-p) - 4 \sum q_{ij}}{p}$$

$$= \sum_{i=1}^p E_i - \frac{(p^2-p) + 4 \sum q_{ij}}{p}$$

$$= (p^2 - p) - 4 \sum q_{ij} - (p^2 - p) + 4 \sum q_{ij} = 0$$

Checking the theorem using example 3.1,

$$\sum_{i=1}^p E_i - \frac{(p^2-p) - 4 \sum q_{ij}}{p} = 1.68818 - 1.7829 + 2.13624 - 2.04152 = 0.$$

**Theorem 3.4**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph. Then

$$\sum_{i=1}^p \varphi_i^2 = (p^2 - p) - 16 \frac{\sum q_{ij}}{p} + 4 \sum d_i^2.$$

**Proof:**

Consider

$$\begin{aligned} \sum_{i=1}^p \varphi_i^2 &= \sum_{i=1}^p \left( E_i - \frac{(p^2 - p) - 4 \sum q_{ij}}{p} \right)^2 \\ &= \sum_{i=1}^p E_i^2 - 2 \left( \frac{(p^2 - p) - 4 \sum q_{ij}}{p} \right) \sum_{i=1}^p E_i + \sum_{i=1}^p \left( \frac{(p^2 - p) - 4 \sum q_{ij}}{p} \right)^2 \\ &= (p^2 - p) - 16 \frac{\sum q_{ij}^2}{p} + 4 \sum d_i^2. \end{aligned}$$

Checking the theorem using example 3.1,

$$\sum_{i=1}^p \varphi_i^2 = 1.68818^2 + (-1.7829)^2 + 2.13624^2 + (-2.04152)^2 = 14.76$$

and

$$(p^2 - p) - 16 \frac{\sum q_{ij}}{p} + 4 \sum d_i^2 = 4(4 - 1) - 16 \frac{\sum q_{ij}}{4} + 4(9.61) = 14.76.$$

**Theorem 3.5**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph. Then

$$\sqrt{p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}} \geq Sle(\zeta).$$

**Proof:**

Cauchy–Schwarz inequality states

$$\left( \sum_{i=1}^n g_i h_i \right)^2 \leq \left( \sum_{i=1}^n g_i^2 \right) \left( \sum_{i=1}^n h_i^2 \right).$$

Assume  $g_i=1$  and  $h_i = |\varphi_i|$  where  $i=1,2, 3 \dots n$

$$\begin{aligned} \text{Then } \left( \sum_{i=1}^p (1) (|\varphi_i|) \right)^2 &\leq \left( \sum_{i=1}^p (1) \right) \left( \sum_{i=1}^p |\varphi_i|^2 \right) \\ \left( \sum_{i=1}^p (|\varphi_i|) \right)^2 &= p \left( \sum_{i=1}^p |\varphi_i|^2 \right) \end{aligned}$$

$$p \left[ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right] \geq Sle(\zeta)^2$$

This implies that

$$\sqrt{p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}} \geq Sle(\zeta).$$

Checking the theorem using example 3.1,

$$\begin{aligned} 7.64884 &\leq \sqrt{4 \left[ 4(4 - 1) - 16 \frac{(3.1)^2}{4} + 4(9.61) \right]} \\ 7.64884 &\leq \sqrt{4(14.76)} = 7.68375. \end{aligned}$$

**Theorem 3.6**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph. Then

$$\sqrt{2 \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}} \leq Sle(\zeta).$$

**Proof:**

$$\begin{aligned} \text{Consider } Sle(\zeta)^2 &= \sum_{i=1}^p |\varphi_i|^2 + 2 \sum_{i < j} |\varphi_i| |\varphi_j| \\ &\geq \sum_{i=1}^p |\varphi_i|^2 + 2 \sum_{i < j} |\varphi_i \varphi_j| \end{aligned}$$

$$\begin{aligned} &= \\ &(p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} + (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \\ &= 2(p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \\ Sle(\zeta) &\geq \sqrt{2 \left[ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right]} \end{aligned}$$

Checking the theorem using example 3.1,

$$\begin{aligned} 7.64884 &\geq \sqrt{2 \left[ 4(4 - 1) - 16 \frac{(3.1)^2}{4} + 4(9.61) \right]} \\ &= \sqrt{2(14.76)} = 5.43323. \end{aligned}$$

**Theorem 3.7**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph and  $\varphi_i = E_i - \frac{(p^2 - p) - 4 \sum q_{ij}}{p}$

Let  $\varphi_{\text{mini}} = \text{mini}|\varphi_i|$  and  $\varphi_{\text{maxi}} = \text{maxi}|\varphi_i|$  where  $1 \leq i \leq p$ . Then

$$\sqrt{p \left[ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right] - \frac{p^2}{4} (\varphi_{\text{maxi}} - \varphi_{\text{mini}})^2} \leq Sle(\zeta)$$

**Proof:**

Using the result [17], Suppose that  $\zeta$  is a graph that has ‘q’ edges and ‘p’ nodes. Then

$$\sum_{i=1}^p g_i^2 \sum_{i=1}^p h_i^2 - \left( \sum_{i=1}^p g_i h_i \right)^2 \leq \frac{p^2}{4} (A_1 A_2 - a_1 a_2)^2$$

where  $A_1 = \text{maxi}(g_i), A_2 = \text{maxi}(h_i),$

$a_1 = \text{mini}(g_i), a_2 = \text{mini}(h_i), 1 \leq i \leq p, i = 1,2.$  Let

$g_i = 1$  and  $h_i = |\varphi_i|.$  Then

$$\sum_{i=1}^p (1) \sum_{i=1}^p |\varphi_i|^2 - \left( \sum_{i=1}^p |\varphi_i| \right)^2 \leq \frac{p^2}{4} (\varphi_{\text{maxi}} - \varphi_{\text{mini}})^2$$

$$p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}$$

$$- Sle(\zeta)^2 \leq \frac{p^2}{4} (\varphi_{\text{maxi}} - \varphi_{\text{mini}})^2$$

$$p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}$$

This implies that

$$\begin{aligned} &\sqrt{p \left[ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right] - \frac{p^2}{4} (\varphi_{\text{maxi}} - \varphi_{\text{mini}})^2} \\ &\leq Sle(\zeta). \end{aligned}$$

Checking the theorem using example 3.1,

$$7.64884 \geq$$

$$\begin{aligned} &\sqrt{4 \left[ 4(4 - 1) + 4(9.61) - 16 \frac{(3.1)^2}{4} \right] - \frac{16}{4} (2.13624 - 1.68818)^2} \\ &= \sqrt{59.04 - 0.80304} = 7.6313. \end{aligned}$$

**Theorem 3.8**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph and  $\varphi_{\text{maxi}}, \varphi_{\text{mini}}$  be same as in Theorem 3.6. Then

$$Sle(\zeta) \geq \frac{2 \sqrt{\varphi_{\text{maxi}} \varphi_{\text{mini}}}}{\varphi_{\text{maxi}} + \varphi_{\text{mini}}} \sqrt{p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}}$$

**Proof:**

Using the result [17],

$$\sum_{i=1}^n g_i^2 \sum_{i=1}^n h_i^2 \leq \frac{1}{4} \left( \sqrt{\frac{A_1 A_2}{a_1 a_2}} + \sqrt{\frac{a_1 a_2}{A_1 A_2}} \right)^2 \left( \sum_{i=1}^n g_i h_i \right)^2$$

Let  $g_i = 1$  and  $h_i = |\varphi_i|$ . Then

$$\begin{aligned} \sum_{i=1}^p (1) \sum_{i=1}^p |\varphi_i|^2 &\leq \frac{1}{4} \left( \sqrt{\frac{\varphi_{maxi}}{\varphi_{mini}}} + \sqrt{\frac{\varphi_{mini}}{\varphi_{maxi}}} \right)^2 \left( \sum_{i=1}^p |\varphi_i| \right)^2 \\ p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\} \\ &\leq \frac{1}{4} \left( \sqrt{\frac{\varphi_{maxi}}{\varphi_{mini}}} + \sqrt{\frac{\varphi_{mini}}{\varphi_{maxi}}} \right)^2 Sle(\zeta)^2 \\ &= \frac{1}{4} \frac{(\varphi_{maxi} + \varphi_{mini})^2}{\varphi_{maxi} \varphi_{mini}} Sle(\zeta)^2 \end{aligned}$$

This implies that

$$Sle(\zeta) \geq \frac{2\sqrt{\varphi_{maxi} \varphi_{mini}} \sqrt{p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}}}{\varphi_{maxi} + \varphi_{mini}}$$

Checking the theorem using example 3.1,

$$\begin{aligned} 7.64884 &\geq \frac{2\sqrt{(2.13624)(1.68818)} \sqrt{4 \left\{ 4(4-1) + 4(9.61) - \frac{16(3.1)^2}{4} \right\}}}{2.13624 + 1.68818} \\ &= \frac{2\sqrt{(2.13624)(1.68818)} \sqrt{4 \left\{ 4(4-1) + 4(9.61) - \frac{16(3.1)^2}{4} \right\}}}{2.13624 + 1.68818} \\ &= 7.63082. \end{aligned}$$

**Theorem 3.9**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph and  $\tau(n)$  is defined as

$$\tau(p) = p \binom{p}{2} \left( 1 - \left( \frac{1}{p} \right) \binom{p}{2} \right) [17].$$

Then

$$\sqrt{p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}} - \tau(n)(\varphi_{maxi} - \varphi_{mini})^2 \leq Sle(\zeta).$$

**Proof:**

Using the result [17],

$|p \sum_{i=1}^n g_i h_i - \sum_{i=1}^n g_i^2 \sum_{i=1}^n h_i^2| \leq \tau(p)(A - a)(B - b)$  in which the conditions  $a \leq g_i \leq A, b \leq h_i \leq B$  are satisfied.

Let  $g_i = h_i = |\varphi_i|, a = b = \varphi_{mini}, A = B = \varphi_{maxi}$ .

This suggests that

$$\begin{aligned} |p \sum_{i=1}^p |\varphi_i|^2 - \sum_{i=1}^p |\varphi_i| \sum_{i=1}^p |\varphi_i| | &\leq \tau(p)(\varphi_{maxi} - \varphi_{mini})^2 \\ p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\} Sle(\zeta)^2 &\leq \\ \tau(p)(\varphi_{maxi} - \varphi_{mini})^2 & \end{aligned}$$

$$\begin{aligned} Sle(\zeta)^2 + \tau(n)(\varphi_{maxi} - \varphi_{mini})^2 \\ &\geq \sqrt{p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}} \\ &\sqrt{p \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}} - \tau(n)(\varphi_{maxi} - \varphi_{mini})^2 \\ &\leq Sle(\zeta) \end{aligned}$$

Checking the theorem using example 3.1,

$$7.64884 \geq$$

$$\sqrt{4 \left\{ 4(4-1) + 4(9.61) - \frac{16(3.1)^2}{4} \right\}} - 4(2.13624 - 1.68818)^2 = \sqrt{59.04 - 0.80303} = 7.631315.$$

**Theorem 3.10**

Assume that  $\zeta$  has ‘q’ edges and ‘p’ nodes and is a fuzzy graph. Then

$$Sle(\zeta) \geq \frac{p \varphi_{maxi} \varphi_{mini} + \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}}{\varphi_{maxi} + \varphi_{mini}}$$

**Proof:**

Using the result [17],

$\sum_{i=1}^n h_i^2 + rR \sum_{i=1}^n g_i^2 \leq (r + R) \sum_{i=1}^n g_i h_i$  in which  $r, R$  are real constants,  $g_i, h_i$  are real constants that are positive such that for each  $I, 1 \leq i \leq p$  the conditions  $rg_i \leq h_i \leq Rg_i$  are satisfied.

Let  $g_i = 1$  and  $h_i = |\varphi_i|, r = \varphi_{mini}, R = \varphi_{maxi}$ . This

implies that  $\sum_{i=1}^p |\varphi_i|^2 + \varphi_{mini} \varphi_{maxi} \sum_{i=1}^p (1)$

$$\begin{aligned} &\leq (\varphi_{mini} + \varphi_{maxi}) \sum_{i=1}^p |\varphi_i| \\ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} &+ p \varphi_{mini} \varphi_{maxi} \\ &\leq (\varphi_{mini} + \varphi_{maxi}) Sle(\zeta) \end{aligned}$$

$$(\varphi_{maxi} + \varphi_{mini}) Sle(\zeta) \geq$$

$$p \varphi_{maxi} \varphi_{mini} + \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}$$

$$Sle(\zeta) \geq \frac{p \varphi_{maxi} \varphi_{mini} + \left\{ (p^2 - p) + 4 \sum_{i=1}^p d_i^2 - \frac{16 \sum q_{ij}^2}{p} \right\}}{\varphi_{maxi} + \varphi_{mini}}$$

Checking the theorem using example 3.1,

$$7.64884 \geq \frac{(4)(2.13624)(1.68818) + 14.76}{2.13624 + 1.68818} = \frac{29.18543}{3.82442} = 7.63133$$

**4. Conclusion**

The energy concept of the  $Sl$  of a graph is enlarged to the  $Sl$  of a fuzzy graph. Also, several properties and limitations are obtained. Fuzzy graphs are included in several  $Sl$  energy bounds calculations, and the results were evaluated using an example. Analogous bounds may be discovered by further research on  $Sl$  energy for a few more types of fuzzy graphs, which will be examined in upcoming studies. Seidal Laplacian energy of a fuzzy graph will be extended to intuitionistic fuzzy graph, Pythagorean fuzzy graph etc.

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